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Place Value and Mathematics for Students with Mild Disabilities: Data and Suggested Practices

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Place value is a phenomenon that has ominous implications for developing number sense and meaning and for using alternative algorithms and alternative representations within whole number arithmetic. For the most part, school programs examine place value at a surface level, with a primary focus on having the student identify or state a number value according to its positional setting. For example, in 146 the student is asked to state the value of the "4" as 40 or as four 10's. Seldom is place value examined for its deep structure potential by incorporating, for example, expanded notation to complete an item such as $2\overline{)100+40+6}$. *Our interest is in the deep structure view of place value models (e.g., Ross, 1986, 1989, 1990), which delved more deeply into selected meanings of place value and their implications. In modeling the work of Ross with a sample of students with mild disabilities, the tasks were constructed from her model and the data are presented following her procedure. This paper presents data from a developmental inquiry of the place value performance of 126 students with mild disabilities on six place value tasks. A discussion of the multiplicity of meanings and activities related to knowing and doing arithmetic with an emphasis on place value is presented.*

Key Words: Arithmetic, Elementary, Learning Disabilities, Instructional Strategies

Judging from the paucity of literature on the topic, the concepts and meanings of place value are among the least stressed in mathematics with all students, including those with disabilities. Place value is presented at a surface level in most situations, and generally involves little more than having students name the place value of a column in a written number. That is, a student is shown a number such as 325, in which the 2 is highlighted and the student is expected to say "tens" or "twenty."

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However, experts (e.g., Baroody, 1990; Fuson, 1990) view place value as a higher-order concept because it formulates our system of number notation and the algorithms that comprise arithmetic.

There is a near total neglect of effort to assist the student in developing a variety of conceptualizations of place value as:

1. the basis of our number system and our arithmetic,
2. the basis for estimation and rounding,
3. a way to construct meaning of alternative representations through the use of symbolic forms of expanded notation,
4. the foundation of alternative algorithms,
5. a foundation of our base-ten system with both whole numbers and decimals,
6. a ratio expressing relationships (e.g., between 10 pennies and one dime, to 100 pennies and one dollar),
7. the conservation of number embedded within alternative representations of a number as would be indicated if the student was shown 56 in the form of five 10s and six 1s and then shown 56 in the form of four 10s and sixteen 1s,
8. the potential to explain decimal relationships relative to the 1's column,
9. a way to interpret the oral and written number system.

Place value is difficult for many students to comprehend until they reach the middle grades (Ross, 1986, 1989, 1990). For example, Jesson (1983) examined the performance of 800 primary-grade through middle-school students' development of place value and found slow, but gradual development to the upper grades.

The literature has long suggested that children who have a poor concept of place value tend to experience difficulties with algorithmic procedures (Ashlock, 1986; Reisman, 1977). Traditional place value instruction that occurs before double-column addition is introduced is not sufficient to help these children. The gap widens as more complex algorithms requiring more conceptual understanding of base-ten numeration systems are introduced. Ross (1990) researched several tasks to determine student understanding about place value. Based on her findings, Ross suggested that teachers need to focus more on two-digit numeration, during which time must be allowed for children to think and create their own number sense. Ross also recommended using problem solving, estimation, and alternative algorithms to teach place value rather than teaching it as an isolated topic.

The primary concerns relative to place value involve its relationship to the operations of arithmetic and the extent to which place value should be taught directly (e.g., Baroody, 1990; Fuson, 1990; Fuson & Briars, 1990; Peterson, Mercer, & O'Shea, 1988) or left to develop intuitively (e.g., Kamii, Lewis, & Livingston, 1993). In their studies of first- and second-grade classrooms in which students were learning about place value, Heibert and Wearne (1992, 1993) contrasted a meanings approach with the conventional textbook approach. At each level, the students in the concept-based classroom performed better than those in the textbook-based program. In part, this may be a function of the textbooks themselves; for example, Fuson (1990) has cited numerous limitations to the textbook treatment of place value. The textbooks give short shrift to place value and fail to give it enough attention to take students beyond the rudimentary levels described by Ross (1990), in which the students simply name the value of a column.

Kamii et al. (1993) compared two groups of students instructed in place value using two different programs. The first group was traditionally instructed; the second group was instructed using a pupil-centered program. The fundamental difference between the two programs was that the pupil-centered program allowed students to invent their own procedures for solving computational and story problems, whereas the students in the traditional group were taught specific rules to solve the computation and story problems. The results indicated that the students in the pupil-centered program had a greater understanding of place value and regrouping in double-column addition.

The authors also examined the extent to which students invented their own algorithms and found evidence of place value utilization. Kari and Anderson (2003) described a classroom approach to place value meanings through the use of problem-solving experiences in which the teacher presented a problem and asked the students to offer a variety of solutions. For example, with a problem such as 11+9, the students offered a variety of solutions as to the relationships between the numbers. Hindy (2003) discussed a variety of ways to develop a sense of place value by presenting fifth-grade students with a problem in which two students each have an amount of money, and the students are asked to state this amount as a multiplication sentence (e.g., each has 3, 2x3). She expanded this to each student having 30, 2x30, 300, 2x200 and 3000, 2x3000. Ultimately, the students developed a number of means by which computational principles are mastered (see Table 1 in Hindy, 2003).

A detailed analysis of student participation in a classroom-focused topic, *the candy factory*, described procedures, outcomes, and alternatives in the development of third-grade students' meanings of place value (Bowers, Cobb, & McLain,1999). The students participated in this single-classroom activity for a period of nine weeks. The activities centered on making packages of candy or repackaging candies. The data consisted of analysis of videotapes, field notes, and interviews prior to and after participation. Students were engaged in simulation activities such as using Unifix cubes as well as computer-based simulations. The arithmetical emphasis was on counting ones, tens, and hundreds to represent place value and incorporating addition and subtraction. Considerable importance was given to student dialogue. Five mathematical practices, ranging from counting to addition and subtraction, comprised the mathematics of the project. Perhaps the most relevant outcome was the differences among individual students as they interpreted and implemented place value meanings.

Knowledge of place in base-ten numeration is necessary for understanding of and success in computation algorithms (Ashlock, 1986; Fuson, 1990; Reisman, 1977; Ross, 1986, 1989, 1990). School-taught procedures encourage children to memorize the digits to nine. Then by adding one more, children continue to count by rote beyond 10 without any concept of the base ten numeration system (Reisman, 1977). In the base-ten numeration system the position of each common digit has an expressed value and its value is relative to other digits (i.e., tens times as great or one tenth the value for each common number in 333).

In a general sense, the power of place value has not been fully explored. For example, an understanding of place value is a necessary foundation for estimation, of which three components seem essential. First is an understanding of estimation. Next is an understanding of multi-digit numerals. And, third is an understanding of the relationship between the words used to express place value and their meaning and physical and symbolic representations.

Estimation is more than guesswork; it is a calculated procedure in which students identify the maximum place value representation in one or more numbers (e.g., the 300 and the 200 in 324+213); identify the relational value sought for those numbers (e.g., do they involve addition, subtraction); and perform mental calculations that are completed from left to right (Lee, 1991), and then, depending on the preciseness of the estimation, round off in columns adjacent to the stipulated place value. Many students with disabilities do not understand that estimation goes beyond guesswork, nor do they understand that it is a calculated procedure. As education moves more toward technology, especially with the hand-held calculator and microcomputer-based computation, the role of estimation will become more paramount as the insertion of data is a left-to-right procedure.

It is important to concentrate on the meaning of multi-digit numerals before focusing on computations with algorithmic processes (Ashlock, 1986). Children must have a concept of the digits “1” to “9” and understand the value of each position in a multi-digit numeral as in a power of 10, 100, or 1000, as the case may be. According to Ashlock, children who have difficulty in mathematics can identify and name place value but generally learn the positions by rote. They cannot combine the digit’s face value and its place value.

Fuson (1990) suggested that young children need to construct relationships between words for a numeral and the marks they represent. She advocated using physical materials for understanding the base-ten system to illustrate the positional factors of multi-digit numerals, and that the focus be understanding, not just the procedures of algorithms.

Jordan and her colleagues (e.g., Hanich, Jordan, Kaplan, & Dick, 2001; Jordan & Hanich, 2000) assessed the mathematical thinking of second-grade students with and without learning disabilities. A component of these studies included place value, one segment of which was constructed based upon the work of Ross (1989). The Jordan and Hanich (2000) study included seven place value items, which consisted of a correspondence activity in which students were asked to count a set of 16 chips and then asked to read the number 16, which was printed on a card. The examiner then pointed to the 6 on the card and asked students to show what that part of the number means using chips. A chip-trading task was also included. Students were given a container of yellow chips and a container of red chips and shown a number written on a card and asked to show the same number with chips. In a general sense, the students were able to count 16, read 16, and specify the meaning of 6 in 16. However, they were less specific in stating the meaning of 1 in 16 and in their performance with the chip-trading activity.

The study by Hanich and colleagues (2001) included (a) a counting and number identification task, (b) a positional knowledge activity, and (c) a digit correspondence activity. In the counting task, the student was provided with 16 chips and asked to count the chips; in the number identification task, the student was shown a number (e.g., 415) and asked to read the number aloud. In the positional knowledge task, the student read a number aloud (e.g., 415) and was then queried about which digits were in the hundreds, ones, and tens place, respectively. There were also two digit-correspondence tasks. In the first, the examiner showed the student a card with the number 16 and then asked the student to use chips to show the value of a part of the number that was circled (e.g., 6, and the student was to show 6 chips); the second task consisted of a standard place-value activity and a nonstandard place-value activity. In the former, the student was shown a card with a number (e.g., 43) along with a picture of 43 squares arranged in a format of four sets of 10 and a set of 3 ones. The examiner indicated there were 43 squares on the paper, circled the 3 and asked the student to draw a circle around the number of squares corresponding to the 3. A second task showed the circles in a format as three sets of 10 and a set of ones. For the final activity, the examiner showed a card with a number (e.g., 26) along with a picture of six groups of 4 stars and one group of 2 stars. The examiner drew a picture around the number 6 and instructed the student to draw a circle around the number of stars representing this number (e.g., 6). Results showed that students had an adequate grasp of counting and number identification but a decided lack of proficiency with digit correspondence in standard and nonstandard formats. Assuming digit correspondence is the basis for understanding the relationship among alternative representations between combinations of symbolic forms (e.g., listening and reading) and nonsymbolic forms (e.g., manipulative and pictorial), it is likely the students will be unable to utilize varying formats of alternative representations or meaningfully utilize the base-ten system.

When provided the results of the outcomes of the inquiry reported in this paper, a sample of both general education and special education teachers in a special project were polled on their responses to the data, as it was their students who participated. They were surprised and confused by the results. Some did not realize that their students did not have an understanding of digit correspondence or positional knowledge even though they could perform rote computational tasks. This reinforces the notion that students often learn mathematics as rote memory tasks, rather than with understanding, particularly the tasks of digit correspondence. Students could count, identify two-digit numerals, and count to 100 by tens, but did not know the value of each digit relative to its position in the numeral. Why should one ask a student to perform operations on these numbers when they have no concept of the value of the number itself? Further, none of the teachers had considered assessing place value to the depths undertaken in this inquiry.

Method

Subjects

The sample consisted of 128 students with mild disabilities enrolled in self-contained special education classrooms and mainstreamed for selected activities. The students were grouped noncategorically according to level of achievement. The students were from primary (*N*=56), intermediate (*N*=44), and junior high (*N*=26) grades. Students were selected from 15 different classrooms in five urban schools in a school district with an enrollment of 44,000 students. The district does not service students by disability type. Rather, students are cross-categorically grouped by level of academic achievement. The district does not permit student data to be extracted from student files.

Procedures

Each student was individually interviewed by a trained examiner. The examiners worked in pairs in performing the alternating functions of recorder and interviewer.

Instrumentation

The instrumentation was influenced by the hierarchical framework of Ross (Ross, 1986, 1989, 1990). It consisted of six primary tasks, each with a range of scoring. The instrument is shown in Figure 1.

Figure 1. Task description for Ross tasks.

Task A: Count Orally by Tens – The students were asked to count by tens as high as they could. If they stopped at one hundred, the examiners asked them if they could count any further.

Level 1: Unable to count orally by tens.
Level 2: Can count orally by tens but only to one hundred.
Level 3: Can count orally beyond one hundred.
Level 4: Not applicable.

Task B: Count 48 Beans Efficiently – The examiner asked the students to count a collection of beans that was partitioned into 10 beans in four cups and 8 loose beans. The examiner told the students that there were 10 beans in each cup and some on the table. The examiner then asked the students, "How many beans are there?"

Level 1: Unable to determine the quantity of beans.
Level 2: Used counting by ones.
Level 3: Used some efficient method of counting such as repeated addition, or multiplication.
Level 4: Not applicable.

Task C: Digit Correspondence of the Beans – After the students counted the number of beans in the above task, the examiner wrote down the number 48 on a sheet of paper. The students were asked first the meanings of the 8 (which was indicated by the examiner pointing to the numeral 8) and then the meaning of the 4 (which again was pointed to by the examiner).

Level 1: The digits had no numerical meaning.
Level 2: The student invented meaning not related to the grouping of tens and ones.
Level 3: The student understood that the whole number represented the whole quantity, but confused or reversed the meaning of the digits.
Level 4: The whole numeral must equal the sum of the quantities of the parts of the objects.

Figure 1. continued

Task D: Conservation of Grouped Numbers – Using the same beans and cups as in the previous tasks, the examiner spilled one of the cups and asked if there were more beans now than there were before.

Level 1: The students did not conserve; they thought that the value of the collection of beans had been changed.

Level 2: The students had to recount the collection to be sure that the amount had not changed.

Level 3: The students knew that the quantity of the group had not changed.

Level 4: Not Applicable.

Task E: Knowledge of Correspondence between Individual Digits and a Collection of Ungrouped Numbers – The examiner laid down 25 tongue depressors before the student. The student was asked to count and then write how many tongue depressors were on the table. If the student wrote the correct number the examiner circled the 5, and asked the student, "Does this the part of the 25 have anything to do with how many sticks you have?" Then the examiner circled the 2 and asked the same question.

Level 1: The digits had no numerical meanings.

Level 2: The student invented meaning not related to the grouping of tens and ones.

Level 3: The student understood that the whole number represented the whole quantity but confused or reversed the meanings of the digits.

Level 4: Whole numeral represents whole quantities of objects. The whole must equal the sum of the parts.

Task F: The Position of the Digits Determines the Value of the Number – The examiner wrote the number 37 on a piece of paper and asked the student to read the number. The examiner then asked the student to point to the tens place and then the ones place. Next the examiner wrote down 84 and asked the student to read this number and asked how many tens.

Level 1: The student could not distinguish between individual digits in a two-digit numeral with respect to the ones digit and tens digit.

Level 2: The student knew that the digits are called ones and tens, but the left/right orientation is not firmly established and may make reversal errors.

Level 3: The student can distinguish between ones and tens, but does not know that the tens digit represents how many tens are in the whole quantity.

Level 4: The student can distinguish which digit is the tens and which is ones, and can determine how many tens are in a two-digit number by inspecting the tens digit.

Results

The results are presented following the procedures of Ross (1989). Percentage of students attaining each level on the six tasks is reported by grade group in Table 1. The table is to be read vertically so the percentages listed for the primary sample under Task A, 32, 48 and 19 round to 100%.

Student performance increased with age, indicating a developmental trend. Greater percentages of junior high students performed at the highest level of competency for each task, as 54% of this group attained the highest level in contrast to 17% of the primary level students. The lowest percentage of students performing at the highest level was found at the primary level. Percentages of highest-order performance increased to the intermediate-level students, with the junior high students performing with the highest percentage. There were two exceptions to this developmental trend, in Tasks B and D. At the intermediate level a greater percentage of students performed at the highest level than the junior high school students on these two tasks.

Table 1
Percent of Students Performing at Each Level

Grade	Performance Level	Task					
		A	B	C	D	E	F
Primary	0	0	7	33	31	34	10
(*N*=56)	1	32	53	38	40	59	28
	2	48	5	3	10	12	27
	3	19	35	12	19	14	17
	4	NA	NA	13	NA	12	17
Intermediate	0	0	0	0	0	0	5
(*N*=44)	1	2	21	27	41	27	4
	2	32	0	2	2	14	32
	3	66	80	27	57	23	18
	4	NA	NA	43	NA	36	41
Junior High	0	0	0	0	0	0	0
(*N*=26)	1	4	8	23	31	15	8
	2	12	19	0	15	19	35
	3	85	73	4	54	27	4
	4	NA	NA	73	NA	39	54

A further examination of the levels of performance on Tasks B and D by the intermediate- and junior high-level students shows that a lower percentage of students in the intermediate group responded at a Level 2 than those in the junior high group (intermediates responding at Level 2 was 0% as opposed to 19.2% at the junior high level on task B and on task D 2.3% of the intermediate-level students responded at Level 2 compared to 15.4% of the junior high students). The percentage of students performing at the highest level for most tasks seldom reached 50%.

On the whole, these results paralleled those of Ross (1989), who studied general education students. That is, higher levels of performance were associated with age and grade level, but even at the upper levels, only a modest percentage of the students attained the highest level of proficiency.

Discussion

Important Considerations

Place value involves several important and underutilized considerations for arithmetic. Among these are (a) the role place value plays in assisting students to enhance their ways of knowing and doing arithmetic; (b) the use of place value with alternative representations; (c) the role of place value in the development and use of alternative algorithms; and (d) the role of place value in the development and conduct of "hands-on" and other forms of assessment. Each of these will be discussed below.

Ways of knowing and doing arithmetic. The most common approach to teaching arithmetic to students with learning disabilities is to present a traditional algorithm accompanied by a set of rules, and to instruct the students to follow an example and complete each item as illustrated in an explicit and rote manner. The result is that the students do not develop knowledge of substantive meanings of the similarities and differences among the operations, nor do they learn that there are many ways of doing the operations. For example, using subtraction as an illustration, students are generally taught that subtraction is "take away." If so, how do students deal with: *Jim has 3 apples in his pail. Nancy has 7 apples in her pail. How many apples must Jim add to his pail to have as many apples as Nancy?* or *Jim has 7 apples in his pail. This is 3 more than he started with. How many apples did Jim start with?*

Actually, subtraction is a search for the difference between two numbers. Sometimes this involves "take away;" at other times this involves other conceptualizations. Also, subtraction is generally taught as a right-to-left operation. whereas the reality is that there are many algorithms for the teaching of subtraction (Cawley & Foley, 2002). For example a left-to-right algorithm is appropriate in the following example where we see the power of the understanding of place value.

$$\begin{array}{r} 45 \\ \underline{-\ 14} \\ 30 \\ \underline{+\ \ \ 1} \\ 31 \end{array}$$

Place value and alternative algorithms. Arithmetic is typically presented to students with learning disabilities via traditional algorithms. That is, the rule for addition, subtraction, and multiplication is to "start with the column on the right." However, that is only one rule! Another strategy may start addition or multiplication in any column as shown below, and clearly going from left-to-right with subtraction has many advantageous (Cawley & Foley, 2002).

Addition. It is time for the Friday Quiz. The students have been working on items such as 235+134=_______. One teacher has prepared a quiz comprised of six items as follows:

1. $\begin{array}{r} 321 \\ \underline{+\ 345} \end{array}$	2. $\begin{array}{r} 441 \\ \underline{+\ 142} \end{array}$	3. $\begin{array}{r} 512 \\ \underline{+\ 355} \end{array}$
4. $\begin{array}{r} 125 \\ \underline{+532} \end{array}$	5. $\begin{array}{r} 414 \\ \underline{+\ 333} \end{array}$	6. $\begin{array}{r} 253 \\ \underline{+\ 115} \end{array}$

The students are instructed to provide the correct answer.

A second teacher has also prepared a quiz of six items.

$$\begin{array}{rrcrrcrr} & \text{ABC} & \qquad & & \text{ABC} & \qquad & & \text{ABC} \\ 1. & 321 & & 2. & 321 & & 3. & 321 \\ & \underline{+\,345} & & & \underline{+\,345} & & & \underline{+\,345} \end{array}$$

$$\begin{array}{rrcrrcrr} & \text{ABC} & \qquad & & \text{ABC} & \qquad & & \text{ABC} \\ 4. & 321 & & 5. & 321 & & 6. & 321 \\ & \underline{+\,345} & & & \underline{+\,345} & & & \underline{+\,345} \end{array}$$

The instructions are for the students to follow the sequence for each item and to provide the correct answer.

Question #1. Start with A, go to B, and then do C.
Question #2. Start with B, got to C, and then do A.
Question #3. Start with C, go to B, and then do A.
Question #4. Start with A, go to C, and then do B.
Question #5. Start with B, go to A, and then do B.
Question #6. Start with C, go to A, and then do B.

Note that the first teacher provided six different items and instructed the students to do them all the same way, whereas the second teacher provided her students with the same item six times and then instructed them to do the item six different ways. The quiz prepared by Teacher A is illustrative of one likely to have been prepared by 99% of teachers and 99% of persons conducting research, where the primary concern is number correct. The quiz prepared by Teacher B is rare and likely to be directed to determining the extent to which the students can demonstrate alternative ways of doing addition and the utilization of place value. Which of these teachers do you believe had a greater interest in higher levels of understanding of addition and place value?

Multiplication. Multiplication is one of the more rotely taught and performed operations of arithmetic, and a key component of this is the emphasis on teaching the tables. We prefer to seek competency in multiplication through array models (Cawley, 2002), where the stress is on meaning rather than memory. This enhances students' capability to utilize alternative algorithms and to stress meaning.

The students are presented with the following multiplication problem and asked to complete the item using the procedure that has been commonly taught.

$$\begin{array}{r} 3\;2\;1 \\ \underline{\text{x}\quad 2} \\ 6\;4\;2 \end{array}$$

After doing so, the instructor presents the students with the following problem and asks them to describe the ways in which the original item and the new item are similar and different.

$$\begin{array}{lr} & 300+20+1 \\ \text{x} & 2 \\ \hline & 2 \\ & 4\,0 \\ + & 6\,0\,0 \\ \hline & 6\,4\,2 \end{array}$$

The students ought to say something to the effect that, "They are the same in that the new one shows the same item except that it is written in the long way." "The big difference between them is the way they are written."

The teacher might then refer to the original problem and go through the steps the students used to complete the item in a manner similar to the approach illustrated in the addition problem. The teacher might say, "You did this by starting here [point to 1s]. Look at this item; see the letters at the top [point to A, B and C]. Can you do this by starting with the number shown by the letter?"

$$\begin{array}{lr} & \text{A B C} \\ & 3\,2\,1 \\ \text{x} & 2 \\ \hline \end{array}$$

And saying, "Watch me, I can start here" and begin with the 10s as marked by the B.

$$\begin{array}{lr} & \text{A B C} \\ & 3\,2\,1 \\ \text{x} & 2 \\ \hline & 4\,0 \end{array}$$

And finally go to the 100s as marked by the A

$$\begin{array}{lr} & \text{A B C} \\ & 3\,2\,1 \\ \text{x} & 2 \\ \hline & 4\,0 \\ & 6\,0\,0 \end{array}$$

and then ask a student if he/she could finish the item by doing C

$$\begin{array}{lr} & \text{A B C} \\ & 3\,2\,1 \\ \text{x} & 2 \\ \hline & 4\,0 \\ & 6\,0\,0 \\ + & 2 \\ \hline \end{array}$$

to show

$$\begin{array}{lr} & \text{A B C} \\ & 3\,2\,1 \\ \text{x} & 2 \\ \hline & 4\,0 \\ & 6\,0\,0 \\ + & 2 \\ \hline & 6\,4\,2 \end{array}$$

With the use of other combinations involving multiplication of three-digit numbers by two-digit numbers (e.g., 321x12,) with or without renaming, the student is able to explore the interrelationships represented by the place value and to ultimately demonstrate and explain what place value does for our understanding of arithmetic, which in this instance is that multiplication can begin and end in many places. This encourages the student to think about mathematics and to develop a sense of number beyond the rote routines typically found in school.

The authors visited a fourth-grade class taught by a general education teacher in a special project. The students were doing lattice multiplication, and one student remarked, "We use different algorithms in this room." Knowing they were using different algorithms was a highly conscious indication of the manner in which meaning was evident in the room. The teacher suggested to the class that the authors might know a different algorithm. The authors presented the problem 321x12 described above. It took no more than 15 minutes for each of the six groups of students to take an individual item and complete it using the A, B, C combinations. The authors explained that the preferred way of doing items was the one they generally used in class, but that the use of different algorithms provided different ways of thinking about math. Thus, the role of place value was stressed.

A special education teacher in this same school and in the same project worked with a self-contained class of 12 students with learning disabilities. The following anecdote was taken from a video of one class session. The students were working with combinations of two- and three-digit addition and subtraction problems where they were encouraged to work both from right to left and left to right. The teacher instructed the students to go beyond the items they were working with in class and to think up items of their own. The teacher then asked various students to go to the board, put an item on the board, and discuss the item.

A student put the following two items on the board:

$$\begin{array}{r} 44 \\ \underline{-\ 23} \end{array} \qquad \begin{array}{r} 44{,}000 \\ \underline{-\ 23{,}000} \end{array}$$

The teacher asked the student if one item (i.e., 44,000-23,000) was harder than the other. The student responded, "No it is not any harder. It is just longer, because you can forget the zeros." This expression of place value sense is common throughout the lessons and prominent when the lesson encourages students to use their own thinking and item development.

Within the community of students with learning disabilities it is common to find students who can say "hundred" or "three hundred" when asked to tell what the 3 in 325 shows. But these same students, when shown three 100s, three 10s, and three1s in a number, such as 333, are not able to affix significant meanings to the item if written as 300+3+3 or explain the comparative value of the 100s, 10s, and 1s, or how to trade one for the other. For example, in one teaching situation involving the authors, a group of eight elementary school-age students with mild disabilities were engaged in a place value activity. The students were shown three cups of different colors (green, blue, and yellow). Each cup had a sticker identifying it as the 100s, 10, or 1s. A number (i.e., 426) was presented to the students, and they were asked to make a representation of the number by putting the proper number of sticks in each cup (4 sticks were placed in the 100s cup, etc.). The students consistently placed the correct number of sticks in the correct cup. The order of the cups was then changed so that the green cup with the 100s sticker was moved to the middle position and the blue cup with the 10s sticker and the yellow cup with the 1s sticker were placed on the extremes. Next, the cups were put in their original positions and the stickers were changed to different cups (e.g., the blue cup, instead of the green cup now had the 100s sticker). The students were totally baffled. Only one out of the eight students was able to relate the actual value of the number to the correct cup or the correct sticker. Yet, all continued to tell what "place value" was represented by a number when it was presented in written format.

One must wonder how students with competence with place value could allow anyone to tell them the "3 does not go into 2" in $3\overline{)246}$, "so we move over." It would seem that students would say something to the effect that the "2 represents 200 and surely 3 goes into 200." Better yet, it seems impossible that teachers continue to tell students that "3 does not go into 2." If the goal is for students with disabilities to develop a "sense about numbers," it seems that greater priority must be given to place value (Foley & Cawley, 2003).

Understanding conservation about numbers is important for both forward (carrying) and backward (borrowing) processing. When students "carry" or "borrow," it is important that they recognize that there is no change in the value represented in the original item, as shown below:

$$\begin{array}{rcl} 43 & = & 40 + 3 \\ \underline{+9} & = & \underline{+ \quad 9} \\ 52 & = & 50 + 2 \end{array} \qquad \begin{array}{rcl} 52 & = & 50 + 2 \\ \underline{-9} & = & \underline{- \quad 9} \\ 43 & = & 40 + 3 \end{array}$$

The value of 43+9 has not changed when represented by 52, nor has the value of 52-9 changed when represented by 43. Students who lack "number sense" are not aware that 40+3+9 and 50+2 represent a common value in that the 1s and the 10s have been renamed, not revalued.

For the most part, our work with place value utilizes a format that is more similar to that of Baroody (1990) than that of Jordan and Hanich (2000), in that we make explicit the depiction of hundred, tens, and ones with sticks or blocks. We wrap 10 popsicle sticks to make a 10, 10 tens to make 100, and so forth. This minimizes the need for students to interpret ratios as in the case of a chip being so much and a number of those chips "ratioed" out to another relationship.

Alternative representations and place value. Within the realm of place value and much of other mathematics is a prevailing concern about manipulatives and other forms of representations (e.g., Kamii et al., 1993; Peterson et al., 1988). The general perspective is that the use of blocks and other materials is helpful in learning about place value, but just how much help is gained from their use is not clear. For example, a research synthesis of mathematics instruction stipulates that "time should not be wasted" (Dixon, n.d., p. 23), and that the use of manipulatives requires more time with larger numbers and, hence, is inefficient.

The view of the present authors is that the significance of the use of objects and related materials associated with manipulation is not clear. For example, some work (e.g., Cawley, Fitzmaurice-Hayes, & Shaw, 1988) makes a clear distinction between the act of manipulation with pictures or objects and the fixed display of pictures or objects. This perspective stipulates that the act of manipulation is an active process in which two-dimensional or three-dimensional items are moved or rearranged. The display is a fixed representation of two-dimensional or three-dimensional items. Manipulation and spoken language are parallel, in that the message conveyed by each is done sequentially (i.e., the acts of manipulation are observed in sequence in much the same manner that words spoken by one person are conveyed to the other in sequence). Also, in both manipulative and spoken language the message fades immediately. An important feature of manipulative and spoken language is that both can be easily reframed or restated. The message in the fixed display remains stable and may be viewed in a holistic or searching form that is not rooted in a specific sequence. Manipulative and spoken messages are memory dependent, and the student must capture and remember the message as it is transmitted. This is not so for display or written symbolic forms, because either can be reviewed exactly as it was originally presented. At the same time, pictorial and written symbolic messages are difficult to modify. Thus, there are important trade-offs in using varying message formats. This tends to have implications for students who must transpose the spoken value to a written value or vice versa (Fuson, 1990). In the teens, we state a value such as 14 by stating the "four" first and the "teen" second; in the thirties, we state a value such as 34 with the "thirty" first and the "four" second. Larger numbers such as 6,534 are stated largely by value, "Six thousand, five hundred, thirty-four," but must be written by positional value. When students are requested to read two numbers composed of three or more digits (i.e., 324+241), they are expected to read from the hundreds to the ones. When writing a number, the students write the number from the highest place value position to the least. The same is true when entering numbers into a calculator or a computer.

Peterson and colleagues (1988) conducted a study of place value learning with a sample of 24 students with learning disabilities. The focus was the use of alternative representations in which one sample used a concrete-semi-concrete-abstract sequence (CSA) and the other sample used only abstract materials. Students in the CSA intervention attained significantly higher scores than those in the abstract intervention. One limitation of the effort was that the terminal objective only asked the students to identify the number of ones or tens in a double-digit number. This is the lowest level of place value use in Ross' hierarchy (Ross, 1990). Our general sense is that one of the reasons for student difficulty with the use of alternative representations is a lack of comprehensive experiences with them. For example, given $3\overline{)246}$, how many students could represent that number with manipulatives?

The term *alternative representations* encompasses a variety of constructs related to the use of three-dimensional (e.g., blocks, sticks) and two-dimensional (e.g., pictorial forms) and spoken and written formats by which number sense is represented and meanings and skills developed. Bruner (1968) used the terms *enactive*, *iconic,* and *symbolic,* and Peterson and colleagues (Peterson et al., 1988) used *concrete*, *semi-concrete*, and *abstract* as forms of depicting alternative representations. Each of these associates a term with material representations in that *enactive* and *concrete* are associated with three-dimensional representations and iconic or semi-concrete are related to two-dimensional representations. One missing factor in each is that of differentiating between spoken and written forms of representations. In our own work (e.g., Cawley, Fitzmaurice-Hayes, & Foley, 2006; Cawley & Reines, 1996), we found it necessary to expand upon the forms of alternative representations and to encompass them into a format that would accentuate their coverage and their reliability. This format is referred to as the Interactive Unit (IU), see Figure 2.

Figure 2. Interactive unit (IU) input and output combinations.

		Input			
		Manipulate	Display	State	Write
	Manipulate	X	X	X	X
	Identify	X	X	X	X
Output	State	X	X	X	X
	Write	X	X	X	X

The IU consists of four channels of input and four channels of output, resulting in a system of 16 interactions by which alternative representations can take place. The four forms of input are manipulate, display, state and write. The four means of output are manipulate, identify, state and write. The term *manipulate* is associated with movements such as arranging, piling, or sequencing. The term *display* stipulates a fixed representation. Either may use three-dimensional or two-dimensional type of materials. If blocks or pictures are moved, this is manipulation. If blocks or pictures are presented in a fixed format, this is display. *State* refers to spoken language and *write* refers to the use of letters, numerals, or other forms of symbolic representations of mathematics. Manipulation and state have common elements, in that both require the student to attend to sequence and, in both instances, the message fades as it is presented. The representation presented by manipulation does not remain before the student, so the student must attend to the action. When manipulation is used, the materials are removed from the view of the student within one or two seconds, and this forces the student to attend to the sequence of steps represented in the *manipulation*. Both manipulation and state invoke an element of short-term memory. An important feature of manipulation and state is that both can be revised quickly by the presenter. Such is not the case with display or write. The write and display options remain fixed before the student and there is an opportunity for the student to review the representations.

Table 2 displays the percent correct for students with mild disabilities performing place value tasks in the write/manipulate interactions. The participating students were from classrooms different from those receiving the Ross tasks. However, they were students in special education from classrooms where the teachers were engaged in a project that stressed the use of alternative representations, alternative algorithms, and meanings. The patterns of response were similar across items, in that the older students performed with greater accuracy than the younger students, except for the first two items on which all students did equally well.

Table 2

Percent of Students Performing at Alternative Representations Utilizing the Interactive Unit Interaction of Write/Manipulate

Problem No.	Problem	Chronological Age				
		9 *(N=24)*	10 *(N=22)*	11 *(N=22)*	12 *(N=28)*	13 *(N=21)*
1	27	95	100	100	100	90
2	356	83	90	100	100	90
3	240	54	86	90	96	85
4	39+18	62	77	95	92	90
5	87-46	58	68	95	92	90
6	306+18	45	59	90	67	76

Figure 3 describes the tasks performed by the students referenced in Table 2.

Figure 3. Task descriptions for alternative representations.

Administrative directions

Say, "See this" [Show flashcard with 6].

Say, "Watch me. I am going to make a representation of six."

[Place 6 sticks on table]

Say, "See, I made a representation six."

Say, "Now, let me see you do one."

Say, "See this" [Show flashcard with 9].

Say, "Take these sticks and make what it shows on the flashcard."

Correct as needed.

Place a set of sticks consisting of units of 1, units of 10 as ten 1s and units of 100 as ten 10s and say, "I want you to use the sticks and make a representation of ..." continue with the following sequence of problems 27, 356, 240, 39+18, 87-46, and 306-180.

Many students are unable to create manipulative representations of simple arithmetic as would be found in $12\overline{)264}$ and as shown below.

The student is presented with a set of popsicle sticks previously grouped into sets of 100s with each 100 comprised of 10 sets of 10; each set of 10 is comprised of ten 1s; a set of 1s is also included. $12\overline{)264}$

The student begins by selecting the correct number of 100s, 10s, and 1s and displaying them as $12\overline{)xxyyyyyyzzzz}$ with x = 100, y = 10, z = 1. (For a more detailed illustration see Foley & Cawley, 2003.)

Once the student has completed the item and illustrated $12\overline{)264}$, the student is instructed to transpose the manipulative representation to the *write* format and show the item in expanded notation form. The student shows $12\overline{)200+60+4}$.

Another student is requested to show the expanded notation format in the traditional form, $12\overline{)264}$ and to explain the similarities between expanded notation and the traditional form.

To illustrate what tasks in other operations would resemble, we provide an example of the Interactive Unit with a display input across four output combinations in Figure 4.

Figure 4. An example of the interactive unit (IU) with a display input and four output options.

Input	Output
Display Teacher uses a pictorial format to present a representation of 233 as: *xx* *yyy* *zzz* where x = 100, y = 10, z = 1	*Manipulate* Student uses a set of sticks to create a representation of 233 as: *xx* *yyy* *zzz* where x = 100, y = 10, z = 1
	Identify Student selects from two or more choices a representation of 233 that corresponds to that displayed by the teacher, such as: A: *xx* *yyy* *zzz* B: *xx* *yyy* *zzz*
	State Student examines the standard presented by the teacher and states the value of the corresponding numbers (i.e., 2 = "200;" 3 = "30;" 3 = "3").
	Write Student examines the standard presented by the teacher and writes the numeral sequence (i.e., 233).

Place value is also important in assisting students to make the transfer from manipulative or pictorial representations to the traditional symbolic representations. Assume students are presented with a pictorial representation of (e.g., these might be dollar bills of different denominations) in the following form:

$$\begin{array}{r} xx \\ \underline{+\ xx} \end{array} \qquad \begin{array}{r} yyy \\ \underline{+\ yy} \end{array} \qquad \begin{array}{r} zz \\ \underline{+\ \ z} \end{array}$$

where $x = 100$, $y = 10$, $z = 1$
and requested to write that number as it is represented in expanded notation (i.e., the long way). The response ought to show:

$$\begin{array}{r} 200 \\ \underline{+\ 200} \end{array} \qquad \begin{array}{r} 30 \\ \underline{+\ 20} \end{array} \qquad \begin{array}{r} 2 \\ \underline{+\ \ 1} \end{array}$$

Assume the student is next asked to write it the short way:

$$\begin{array}{r} 233 \\ \underline{+\ 221} \end{array}$$

The transposition from pictorial representations is most commonly undertaken by presenting the pictorial form and then moving directly to the "short form." However, many students do not make the transition to the "short form" because they lose the meaning between the three hundreds when represented pictorially and when written symbolically. The transposition can be made more understandable with the assistance of expanded notation. Understanding place value is fundamental in making the transposition from manipulative or pictorial representations to expanded notation and then to the "short form." It is also essential for reversing the procedure when going from the short form to expanded notation to a manipulative or pictorial representation to assess student performance. It is also important to consider output as an equivalent partner in the exchange of information. In some of our work (Foley, Parmar, & Cawley, 2004) we have stressed the role of manipulative outputs such as would be the case in the actual construction of an aircraft carrier following the written and pictorial specifications of architects and engineers. At a simpler level, alternative representations apply mathematics activities to daily life, such as when one goes shopping. The person reads the shopping list and finds the product (e.g., a can of peas), which is then taken from the shelf and placed in the shopping cart (i.e., write/manipulate). The procedure is reversed as the cashier takes the product and scans it to determine a symbolic price.

In summary, the role of place value has numerous implications beyond positional knowledge. Program and material development specialists should extend the role of place value in their work. Teachers should expand upon the many dimensions of place value and interpret its value within the context of "number sense" and go beyond positional knowledge.

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